APPLICATION NOTE

> **DID YOU KNOW ?**
> Albert Einstein is said to have once been asked if he wanted to be part of the government of the new nation of Israel. As the story goes, he replied that politicians are forgotten, but equations are forever. More precisely, he said:
> "Yes, we have to divide up our time like that, between our politics and our equations. But to me our equations are far more important, for politics are only a matter of present concern. A mathematical equation stands forever."

Op-Amp Errors, Another View

Preamble

The subject of Op-Amp errors has been covered by many writers. Dataforth's Application Note 102 [1] covers the topic as applied to instrumentation amplifiers and provides links to spreadsheets for worst case error analysis. This is a good place to go for the practical engineer. The instrumentation amplifier of Application Note 102 can be decomposed into the non-inverting amplifier of this application note using Bartlett's bisection theorem [2].

Using circuit simulators and suitable Op-Amp macros will provide quick results. Two free downloads are TI Tina and LTspice. These may be found on the websites of Texas Instruments and Linear Technology.

This note concerns only Op-Amps and does not attempt to be exhaustive. Its objective is to show how final equations of some basic errors are derived from first circuit principles – primarily node analysis. Final equations are enclosed with a box.

This Application Note is for the math driven reader. We have all had the experience in our math and technical education of reading a treatment of some topic where the author introduces material and jumps to a final equation stating "the derivation is left to the interested reader." Unless our teacher gave the derivation as an assignment, most of us were not interested. Another hated statement is "may be seen by inspection." Not all students are interested inspectors. It does have to be admitted that many books would be much too long if not for these shortcuts. It is also true that self-taught knowledge, while frequently incomplete, often stays longer. Perhaps non-math oriented people never knew that following along with pencil and paper is what makes math come alive. For many, math becomes a blur of meaningless symbols, mindless formula plugging, or even blind imitation.

An elegant way to do the following analysis is using Middlebrook's Extra Element Theorem [3]. However, I start with basic circuit analysis and turn the algebraic crank.

Offset Voltage

Assume the circuit of Figure 1 is the "Ideal Operational Amplifier" except for offset voltage and finite open loop gain.
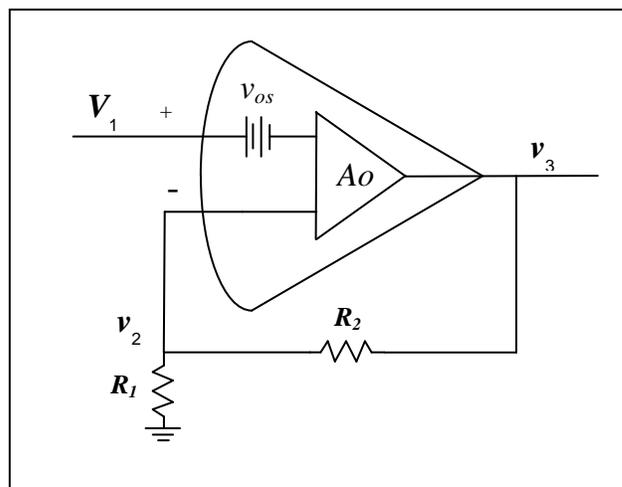

Figure 1
Offset Voltage

First, the ideal amplifier. For A_o to be very, very large, the independent variable V_1 must be close enough, for all practical purposes, to the dependent variable v_2. To find the Ideal Amplifier Gain, use the voltage divider relation:

$$v_3 \frac{R_1}{R_1 + R_2} = v_2 = V_1 \ thus \ \frac{v_3}{V_1} = \frac{R_1 + R_2}{R_1} = 1 + \frac{R_2}{R_1} = Ideal \ Amplifier \ Gain \ (G_I)$$

Negative feedback amplifiers are said to have closed loop gain (G_{cl}). When the feedback connection is not made, the phrase becomes open loop gain (G_{ol}). $\frac{R_1}{R_1+R_2}$ is called the feedback factor. In this note β symbolizes the feedback factor.

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$$G_I = 1 + \frac{R_2}{R_1} = Ideal\ Closed\ Loop\ Gain\ for\ A_o \to \infty$$

$$\beta = \frac{1}{G_I} = \frac{R_1}{R_1 + R_2} = Feedback\ Factor$$

Eq. 1.1 $$v_3 \triangleq Ao[(V_1 - v_{os}) - v_2]$$

Eq. 1.2 $$\frac{v_2}{R_1} + \frac{v_2 - v_3}{R_2} = 0$$

Rearranging the terms,

Eq. 1.1a $$v_3 = A_o(V_1 - v_{os}) - A_o v_2$$

$$v_2 = \frac{v_3 - A_o(V_1 - v_{os})}{-A_o} = \frac{A_o(V_1 - v_{os}) - v_3}{A_o}$$

$$v_2 = V_1 - v_{os} - \frac{v_3}{A_o}$$

Eq. 1.2a $$v_2\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{v_3}{R_2}$$

$$v_2 = \frac{R_1}{R_1 + R_2} v_3 = \beta v_3$$

Combining Eq. 1.1a and Eq. 1.2a,

$$V_1 - v_{os} - \frac{v_3}{A_o} = \beta v_3$$

$$V_1 - v_{os} = \beta v_3 + \frac{v_3}{A_o} = v_3\left(\beta + \frac{1}{A_o}\right)$$

$$v_3 = \frac{V_1 - v_{os}}{\beta + \frac{1}{A_o}} \to \frac{V_1 - v_{os}}{\beta}\ for\ A_o \to \infty$$

Eq. 1.3

$$v_3 = \frac{V_1 + v_{os}}{\frac{R1}{R_1 + R_2} + \frac{1}{A_o}} = \frac{(V_1 - v_{os})\frac{1}{\beta}}{1 + \frac{1}{\beta A_o}} = \frac{G_I(V_1 - v_{os})}{1 + \frac{G_I}{A_o}}$$

Equation 3 easily, perhaps eloquently, shows how offset voltage and limited open loop gain modify the familiar ideal gain equation. Modern Op-Amps have very high open loop gain. If $A_o \to \infty$, the error of offset voltage alone is easily observed. It is multiplied by the closed loop, ideal gain. This is often the most serious problem in high gain Op-Amp applications.

Note that this is really basic feedback theory and applies to much more than just Op-Amps. Some communication amplifiers and various control problems are examples with open loop gain far from infinite.
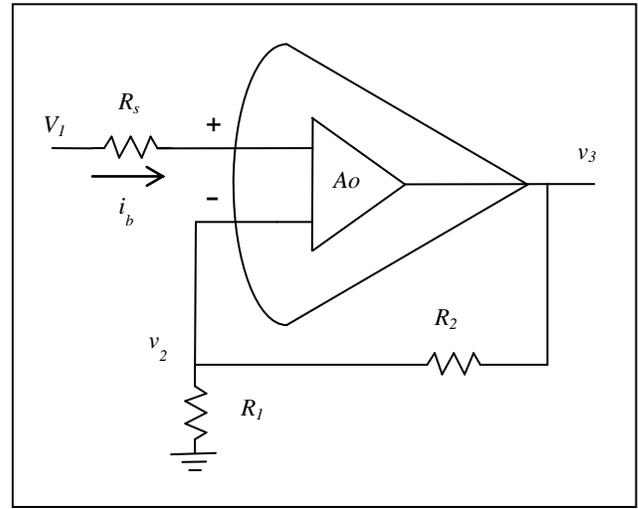
Bias and Offset Current

Figure 2
Bias Current

Define $$\beta = \frac{R_1}{R_1 + R_2} = \frac{1}{Closed\ Loop\ Gain} = \frac{1}{G_I}$$

Eq. 2.1 $$v_3 \triangleq A_o(V_1 - R_s i_b - v_2)$$

Eq. 2.2 $$\frac{v_2}{R_1} + \frac{v_2 - v_3}{R_2} = 0$$

Eq. 2.1a $$A_o v_2 = A_o(V_1 - R_s i_b) - v_3$$

$$v_2 = V_1 - R_s i_b - \frac{v_3}{A_o}$$

Eq. 2.2a $$v_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{v_3}{R_2}$$

$$\frac{v_2}{R} = \frac{v_3}{R_2}, \quad where\ \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}, \quad \xrightarrow{yields}\ R = \frac{R_1 R_2}{R_1 + R_2} = \beta R_2$$

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Eq. 2.1a and 2.2a

$$v_3 \frac{R}{R_2} = V_1 - R_s i_b - \frac{v_3}{A_o}$$

$$v_3 \left(\frac{R}{R_2} + \frac{1}{A_o}\right) = V_1 - R_s i_b$$

$$v_3 = \frac{V_1 - R_s i_b}{\frac{R}{R_2} + \frac{1}{A_o}} = \frac{V_1 - R_s i_b}{\frac{R_1}{R_1 + R_2} + \frac{1}{A_o}}$$

$$v_3 = \frac{V_1 - R_s i_b}{\beta + \frac{1}{A_o}} = \frac{\frac{1}{\beta}(V_1 - R_s i_b)}{1 + \frac{1}{\beta A_o}}$$

Eq. 2.3

$$v_3 = \frac{G_I(V_1 - R_s i_b)}{1 + \frac{G_I}{A_o}}$$

$For\ A_o \rightarrow \infty, Error \rightarrow -G_I R_s i_b$

$Suggestion: Make\ \dfrac{R_1 R_2}{R_1 + R_2} = R_s\ for\ minimum\ error$

For many modern Op-Amp applications, the ideal model is adequate. When it is not, consideration of offset voltage and input current suffices for many of the rest. Exceptions are high gain circuits and power amplifiers driving a heavy load. Note that integrated circuit Op-Amps have almost equal bias currents on the positive and negative inputs. The difference is sometimes called offset or difference current. If the source resistances as seen from each input are nearly the same, the error is minimized.

The rest of this paper is for the interested reader. It has more importance in general control systems.

Output Resistance

The following amplifier is ideal in every way, except it has limited open loop gain and a non-zero output resistance. As we will see, if we assume infinite open loop gain, the output resistance would have no effect. Since we are considering non-zero output resistance, a load resistance is included. Of course, the gain determining resistors are also a load on the amplifier.
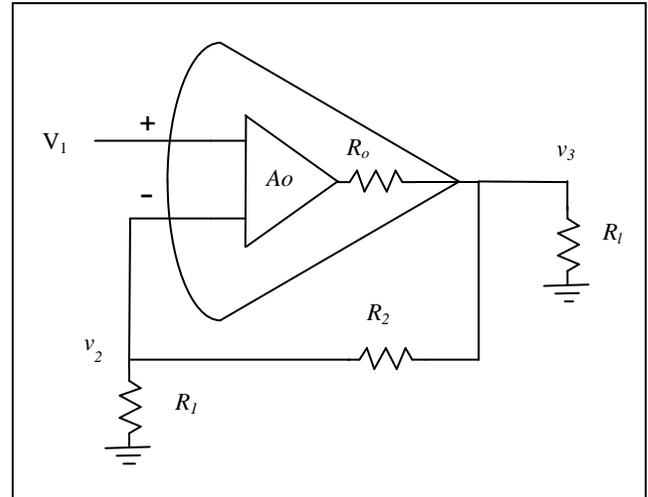

Figure 3
Output Resistance

$$G_I = \frac{v_3}{V_1} = 1 + \frac{R_2}{R_1} = \frac{R_1 + R_2}{R_1} = \frac{1}{\beta} = 1/Feedback\ Factor$$

Eq. 3.1 $\qquad \dfrac{v_2}{R_1} + \dfrac{v_2 - v_3}{R_2} = 0$

Eq. 3.2 $\qquad \dfrac{v_3 - A_o(V_1 - v_2)}{R_o} + \dfrac{v_3 - v_2}{R_2} + \dfrac{v_3}{R_l} = 0$

Gathering terms,

Eq. 3.1a $\qquad v_2 \left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right) = \dfrac{v_3}{R_2}$

Eq. 3.2a

$$v_2 \left(\frac{A_o}{R_o} - \frac{1}{R_2}\right) + v_3 \left(\frac{1}{R_o} + \frac{1}{R_2} + \frac{1}{R_l}\right) = \frac{A_o V_1}{R_o}$$

Let, $\qquad \dfrac{1}{R_{12}} = \dfrac{1}{R_1} + \dfrac{1}{R_2} = \dfrac{R_1 + R_2}{R_1 R_2}$

$$\frac{1}{R} = \frac{1}{R_o} + \frac{1}{R_2} + \frac{1}{R_l} = \frac{R_2 R_l + R_o R_l + R_o R_2}{R_o R_2 R_l}$$

Then, $\qquad v_2 = \dfrac{R_{12} v_3}{R_2}$

Combining equations,

$$v_3 \frac{R_{12}}{R_2} \left(\frac{A_o}{R_o} - \frac{1}{R_2}\right) + \frac{v_3}{R} = \frac{A_o V_1}{R_o}$$

Or

$$v_3 \left[\frac{R_1}{R_1+R_2} \frac{A_o}{R_o} - \frac{R_1}{R_2(R_1+R_2)} + \frac{1}{R} \right] = \frac{A_o V_1}{R_o}$$

$$v_3 \left(\frac{\beta A_o}{R_o} - \frac{\beta}{R_2} + \frac{1}{R} \right) = \frac{A_o V_1}{R_o}$$

$$v_3 \left(\beta - \frac{\beta R_o}{A_o R_2} + \frac{R_o}{A_o R} \right) = V_1$$

$$v_3 \left[\beta + \frac{1}{A_o} \left(\frac{R_o}{R} - \frac{\beta R_o}{R_2} \right) \right] = V_1$$

$$\frac{v_3}{V_1} = \frac{1}{\beta + \frac{1}{A_o} \left(1 + \frac{R_o}{R_2} + \frac{R_o}{R_l} - \frac{\beta R_o}{R_2} \right)}$$

$$\frac{v_3}{V_1} = \frac{1}{\beta + \frac{1}{A_o} \left[1 + R_o \left(\frac{1-\beta}{R_2} + \frac{1}{R_l} \right) \right]}$$

$$\frac{1-\beta}{R_2} = \frac{1 - \frac{R_1}{R_1+R_2}}{R_2} = \frac{R_1+R_2-R_1}{R_2(R_1+R_2)} = \frac{1}{R_1+R_2}$$

$$\frac{v_3}{V_1} = \frac{1}{\beta + \frac{1}{A_o} \left[1 + R_o \left(\frac{1}{R_1+R_2} + \frac{1}{R_l} \right) \right]}$$

$$\frac{v_3}{V_1} = \frac{1}{\beta + \frac{1}{A_o} \left[1 + \frac{R_o}{R_L} \right]} \quad where \quad \frac{1}{R_L} = \frac{1}{R_1 + R_2} + \frac{1}{R_l} = \frac{1}{Total\ Load}$$

Eq. 3.3

$$\frac{v_3}{V_1} = \frac{\frac{1}{\beta}}{1 + \frac{1}{\beta A_o}\left(1 + \frac{R_o}{R_L}\right)} = \frac{G_I}{1 + \frac{G_I}{A_o}\left(1 + \frac{R_o}{R_L}\right)}$$

Equation 3.3 provides a compact view of what R_o does to the gain. Infinite A_o makes it have no effect and reduces the gain to $1/\beta$, as we know it must for an ideal amplifier. Whatever the effect, it may be reduced further by large R_L. With IC Op-Amps, the effect may be exceedingly negligible.
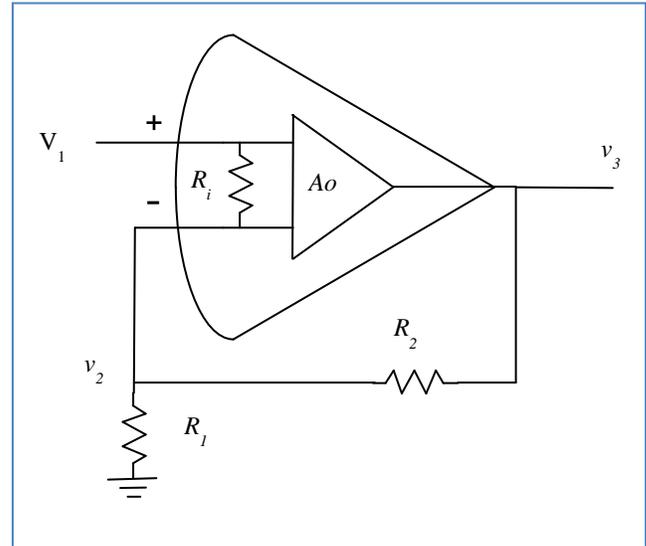
Input Resistance

Figure 4
Input Resistance

Eq. 4.1 $A_o(V_1 - v_2) \triangleq v_3$

Eq. 4.2 $\dfrac{v_2}{R_1} + \dfrac{v_2 - V_1}{R_i} + \dfrac{v_2 - v_3}{R_2} = 0$

$$v_2 \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_i} \right) - \frac{V_1}{R_i} - \frac{v_3}{R_2} = 0$$

Let, $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_i}$

Rearranging terms,

Eq. 4.2a $\dfrac{v_2}{R} - \dfrac{V_1}{R_i} - \dfrac{v_3}{R_2} = 0$

From Eq. 4.1,

$$-A_o v_2 = v_3 - A_o V_1$$

Eq. 4.1a $v_2 = V_1 - \dfrac{v_3}{A_o}$

Combining equations 4.1a and 4.2a,

$$\frac{V_1}{R} - \frac{v_3}{R A_o} - \frac{V_1}{R_i} - \frac{v_3}{R_2} = 0$$

$$V_1 \left(\frac{1}{R} - \frac{1}{R_i} \right) = v_3 \left(\frac{1}{R_2} + \frac{1}{A_o R} \right)$$

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$$\frac{v_3}{V_1} = \frac{\frac{1}{R}-\frac{1}{R_i}}{\frac{1}{R_2}+\frac{1}{A_oR}} = \frac{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_i}-\frac{1}{R_i}}{\frac{1}{R_2}+\frac{1}{A_oR}} = \frac{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}{1+\frac{R_2}{A_oR}}$$

$$\frac{v_3}{V_1} = \frac{1+\frac{R_2}{R_1}}{1+\frac{R_2}{A_oR}} = \frac{G_I}{1+\frac{1}{A_o}\frac{R_2}{R}} \quad where \quad \frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_i}$$

Eq. 4.3

$$\frac{v_3}{V_1} = \frac{G_I}{1+\frac{1}{A_o}\left(1+\frac{R_2}{R_1}+\frac{R_2}{R_i}\right)} = \frac{G_I}{1+\frac{1}{A_o}\left(G_I+\frac{R_2}{R_i}\right)} = \frac{G_I}{1+\frac{G_I}{A_o}\left(1+\frac{R_2}{G_IR_i}\right)}$$

From equation 4.3, we can easily see how R_i changes the "ideal gain." For infinite A_o, R_i has no effect whatsoever. For that reason, we have to consider both in the same model. This is intuitively satisfying from merely looking at Figure 4. If we let R_i approach infinity, we get the effect of A_o alone.

Now we can consider the input resistance of the entire circuit.

Eq. 4.4

$$R_{cr} \triangleq \frac{V_1}{i_1}$$

where R_{cr} is the input resistance of the circuit

Eq. 4.5　　　　$v_3 \triangleq A_o(V_1 - v_2) \triangleq A_o i_1 R_i$

Combining Eq. 4.4 and Eq. 4.5,

$$R_{cr} = \frac{V_1 A_o R_i}{v_3} = \frac{A_o R_i}{v_3/V_1} = \frac{A_o R_i}{gain}$$

Using equation 4.3,

$$R_{cr} = \frac{A_o R_i\left[1+\frac{1}{A_o}\left(G_I+\frac{R_2}{R_i}\right)\right]}{G_I} = \frac{A_o R_i + R_i G_I + R_2}{G_I}$$

$$R_{cr} = R_i\left(1+\frac{A_o}{G_I}\right)+\frac{R_2}{G_I} = R_i + R_i\left(\frac{A_o}{G_I}+\frac{R_2}{R_i G_I}\right)$$

Eq. 4.6

$$R_{cr} = R_i\left[1+\frac{1}{G_I}\left(A_o+\frac{R_2}{R_i}\right)\right]$$

For $A_o \gg 0$, $R_{cr} \cong \frac{A_o}{G_I}R_i$

From this we can see that the input resistance of the Op-Amp is increased by the ratio of open loop gain to the closed loop, "ideal gain." For example, using typical values $10^5/10^2 = 10^3$ gives an increase multiplier of 1000. Note that "ideal gain" is the inversion of the feedback factor:

$$\beta = R_1/(R_1 + R_2) = 1/(1 + R_2/R_1)$$

Some authors often use β, rather than a symbol for "ideal gain."

Input, Output, and Load Resistance Altogether
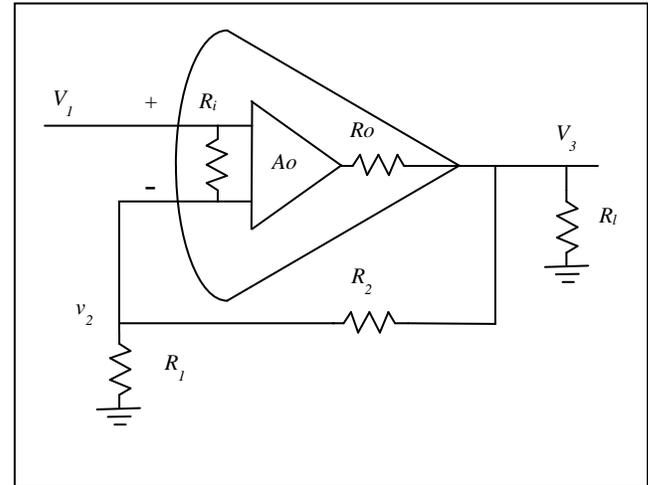

Figure 5
Input and Output Resistance with Load

$$\beta \triangleq \frac{R_1}{R_1 + R_2} = 1/_{Ideal\ Gain}$$

Eq. 5.1　　　$\dfrac{v_3-v_2}{R_2}+\dfrac{v_3}{R_l}+\dfrac{v_3-A_o(V_1-v_2)}{R_o}=0$

Eq. 5.2　　　$\dfrac{v_2}{R_1}+\dfrac{v_2-v_3}{R_2}+\dfrac{v_2-V_1}{R_i}=0$

Eq. 5.1a

$$v_2\left(-\frac{1}{R_2}+\frac{A_o}{R_o}\right)+v_3\left(\frac{1}{R_2}+\frac{1}{R_l}+\frac{1}{R_o}\right)=\frac{A_oV_1}{R_o}$$

Eq. 5.2a　　　$v_2\left(\dfrac{1}{R_1}+\dfrac{1}{R_2}+\dfrac{1}{R_i}\right)-\dfrac{v_3}{R_2}=\dfrac{V_1}{R_i}$

Define:

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_i}; \quad \frac{1}{R_b}=\frac{1}{R_2}+\frac{1}{R_l}+\frac{1}{R_o}$$

Eq. 5.2b $\dfrac{v_2}{R} = \dfrac{V_1}{R_i} + \dfrac{v_3}{R_2}$

Eq.5.1a and 5.2b

$$\left(\frac{V_1 R}{R_i} + \frac{v_3 R}{R_2}\right)\left(\frac{A_o}{R_o} - \frac{1}{R_2}\right) + \frac{v_3}{R_b} = \frac{A_o V_1}{R_o}$$

$$V_1\left(\frac{R A_o}{R_i R_o} - \frac{R}{R_i R_2} - \frac{A_o}{R_o}\right) + v_3\left(\frac{R A_o}{R_2 R_o} - \frac{R}{(R_2)^2} + \frac{1}{R_b}\right) = 0$$

Now, we have a clear relation between the dependent variable v_3 and the independent variable V_1. At this point, I will take the easy way out and say the next intermediate steps are "left to the interested reader." The same algebraic methods of factoring and cancelling used above will get you to Eq. 5.3. Remember as you go that $A_o, R_i, and\ R_l$ must be allowed to clearly approach infinity. So put them in denominators so those terms will approach zero. R_o must approach zero, so put it in numerators. Finally look for any way that negative terms can cancel out.

Finally,

$$\frac{v_2}{V_1} = \frac{\dfrac{1}{\beta} + \dfrac{1}{A_o}\dfrac{R_o}{R_i}}{1 + \dfrac{1}{A_o}\left[\dfrac{1}{\beta} + \dfrac{R_2}{R_i} + R_o\left(\dfrac{1}{\beta R_l} + \dfrac{1}{R_1} + \dfrac{1}{R_i} + \dfrac{R_2}{R_l R_i}\right)\right]}$$

Bringing $1/\beta = G_I$ out to show G_I/A_o,

> **Eq. 5.3**
>
> $$\frac{v_3}{V_1} = \frac{G_I + \dfrac{R_o}{A_o R_i}}{1 + \dfrac{G_I}{A_o}\left\{1 + \dfrac{R_o}{R_l} + \dfrac{1}{G_I}\left[\dfrac{R_2}{R_i} + R_o\left(\dfrac{1}{R_1} + \dfrac{1}{R_i} + \dfrac{R_2}{R_l R_i}\right)\right]\right\}}$$

Again the G_I/A_o ratio plays an important role. $R_o/(A_o R_i)$ in the numerator is clearly insignificant.

From equation 5.3, you may easily see how the amplifier imperfections can go to their ideal limits leaving the ideal gain equation, and there are no vexatious negative signs. Equations with a negative sign cause me to wonder if the whole thing could go negative. Then I have to ask myself if that is physically meaningful or possible. If not, I may have made a mistake. Last of all, don't forget unit analysis. Gain equations must have all terms unit-less, voltages volts, resistances resistance, and so on. This is often a quick way to see something is wrong, or going wrong, in a derivation.

> To generalize on all of the previous, the ratio of G_I/A_o makes much of the difference as to whether these error sources are significant.

Example and Conclusion

Let's see if any of this is worth considering. Consider the following design.

A_o = 200k
R_i = 50kohm
R_o = 100
R_2 = 99.9k
R_1 = 100
R_l = 2k

These numbers in equation 5.3 yield a gain of 994.76, a 0.52% gain error. Almost all of this comes from the denominator, and there only from A_0. A very slight amount comes from the ratio R_o/R_l. I believe we could also calculate an error for one element at a time and assume superposition. Beyond this 0.5% error, the resistor tolerances are obviously important. Offset voltage and bias current may produce larger errors and should usually be considered first. Additional error sources are noise, power supply rejection, and common mode rejection. Noise is a whole subject in itself. Then comes temperature stability and ageing. Room temperature errors can be calibrated out, but temperature and ageing cannot, unless we use some type of reference and do automatic calibration with a feedback loop. If we don't have a processor for this, a clocked feedback integrator with an analog multiplexer can be used for offset error. Just provide feedback to the ground connection of R_1 with an inverting integrator. Be careful of noise.

Perhaps for day-to-day work it is best to just use a good simulation! However, always a however, while quick numerical modeling is a great aid, we easily lose insight into the sources of error. Equations like 5.3 can show where the bulk of error comes from for a particular application. In the example, almost all of the gain error comes from the limited open loop gain and resistor tolerances.

References

1. Dataforth Application Note AN102
   http://www.dataforth.com/catalog/pdf/an102.pdf

2. Bi-section theorem: Bartlett, AC, "An extension of a property of artificial lines," Phil. Mag., Vol. 4, P902, Nov. 1927

3. Middlebrook's Extra Element Theorem: Vorérian, Vatché, "Fast analytical techniques for electrical and electronic circuits," Cambridge University Press, 2002, pp61-106, ISBN 0-521-62422-8